MATHEMATICS

ON THE NUMBER OF ZEROS OF GENERAL EXPONENTIAL POLYNOMIALS

BY

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In this paper it is our aim to find upper bounds for the number of zeros in an arbitrary disk in the complex plane for general exponential polynomials, i.e. sums of the form

$$f(z)=\sum_{k=1}^{l} P_k(z)\, e^{\omega_k z}, \qquad f\neq 0,$$

where ω_k is a complex number and P_k is a polynomial of degree ϱ_k-1 for $k=1, 2, \ldots, l$.

In 1960 P. TURÁN, [18], gave such an upper bound for the case that the P_k's are complex constants. He proved that the number, N_L, of zeros of such a function in a square of side-length L does not exceed

$$6L\Delta+l\log\left(2+\frac{l^3}{\delta L}\right)+\log 2l$$

where $\Delta=\max\limits_{k}|\omega_k|$ and $\delta=\min\limits_{k\neq j}|\omega_k-\omega_j|$.

Four years later S. DANCS and P. TURÁN, [3], gave an upper bound for N_L which is also valid if the P_k's are arbitrary polynomials. They proved for this general case

$$N_L\leqq 9p^2\log 2lp+5lp+L\Delta\left(5+\frac{p}{10l}\right)+p(2l+p)\log\left(1+\frac{5l^2p}{L\delta}\right),$$

where $p=\max\limits_{k}\varrho_k$ and Δ, δ are defined as above. Afterwards similar results were obtained by J. COATES, [2], and A. J. VAN DER POORTEN, [12].

As Dancs and Turán remark in their paper, the point of the theorem is that the upper bound is independent of the coefficients of the polynomial, of the position of the square and only very loosely dependent on the configuration of the ω_k-exponents. Furthermore, easy examples show that the dependence on L, Δ, l and p is indispensable.

In this paper we shall show that the dependence of the upper bound on δ is superfluous. More precisely, we shall prove that the number, N_R, of zeros of f in a disk of radius R cannot exceed

$$3(n-1)+4R\Delta,$$

where $n=\sum\limits_{k=1}^{l}\varrho_k$. (Compare the corollary of theorem 1).

This upper bound is sharper than the estimate due to Dancs and Turán. In the case that $R\Delta$ is small with respect to n, we can deduce two better estimates:

Theorem 2: If $R\Delta \leq \min(n^{-2}, 24^{-2})$, then $N_R \leq n-1$.
The last inequality cannot be improved.

Theorem 3: Let $\varepsilon>0$ and $R\Delta/n < \exp(-6/\varepsilon)$.
Then

$$N_R \leqq n(1+\varepsilon).$$

This last result improves a lemma of A. O. GELFOND, [6] Ch. III, § 4, lemma III. On applying this inequality instead of Gelfonds lemma, one can simplify some theorems on transcendency. We shall return to this question in another paper. See also a paper of R. SPIRA, [14].

Finally, we treat the case that $R\Delta$ is large with respect to n. Then we have:

Theorem 4: $N_R \leqq eR\Delta + n\left(5+\log\dfrac{R\Delta}{n}\right)$ if $R\Delta \geqslant n/e$.

This estimate will be compared with classical asymptotic results due to G. PÓLYA, [10, 11] and E. SCHWENGELER, [13]. See also D. G. DICKSON, [4].

There is an extensive literature on related questions. We only mention papers of R. E. LANGER, [8], H. WITTICH, [19, 20], R. BELLMAN and K. L. COOKE, [1] Ch. XII, D. G. DICKSON, [5], K. MAHLER, [9], R. TIJDEMAN, [15] Ch. VI and VII.

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1. Let z_0 be a complex number and R a non-negative real number. The closed disk with centre z_0 and radius R, i.e. $\{z\colon |z-z_0| \leqq R\}$, is denoted by $C(z_0, R)$. By $N(z_0, R, f)$ we denote the number of zeros of f in $C(z_0, R)$, multiple zeros counted in accordance with their multiplicities. The maximum of $|f|$ on $C(z_0, R)$ is denoted by $M(z_0, R, f)$.

We start with a consequence of the well-known theorem of Jensen:

Lemma 1. Let R, s, t be positive numbers, $s>1$, and let $f \neq 0$ be a function holomorphic on $C(0, (st+s+t)R)$.

Then

$$N(0, R, f) \leqq \frac{1}{\log s} \log \frac{M(0, (st+s+t)\,R, f)}{M(0, tR, f)}. \tag{1.1}$$

Proof. Suppose that z_* is a number in $C(0, tR)$ such that

$$|f(z_*)| = M(0, tR, f). \tag{1.2}$$

Then

$$C(0, R) \subset C(z_*, (1+t)\,R) \tag{1.3}$$

and

$$C(z_*, (st+s)\,R) \subset C(0, (st+s+t)\,R). \tag{1.4}$$

By Jensen's theorem, see e.g. [17] p. 125, we have

$$\int_0^{sR} \frac{N(z_*, r, f)}{r}\, dr = \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_* + sRe^{i\theta})| d\theta - \log |f(z_*)|.$$

Now

$$\int_0^{sR} \frac{N(z_*, r, f)}{r}\, dr \geqq N(z_*, R, f) \log s.$$

Hence by the two preceding formulas

$$N(z_*, R, f) \leqq \frac{1}{\log s} M\left(z_*, sR, \log \frac{|f|}{|f(z_*)|}\right).$$

This implies by (1.2)–(1.4)

$$N(0, R, f) \leqq N(z_*, (1+t)\,R, f) \leqq \frac{1}{\log s} \log M\left(z_*, sR(1+t), \frac{f}{f(z_*)}\right) \leqq$$

$$\leqq \frac{1}{\log s} \log \frac{M(0, (st+s+t)\,R, f)}{M(0, tR, f)},$$

as asserted by the lemma.

2. In the remainder of this paper we shall use the following notations:

(2.1) Let l be a positive integer, $\omega_1, \omega_2, \ldots, \omega_l$ complex numbers, and let $P_1, P_2, \ldots, P_l$ be non-trivial polynomials of the respective degrees $\varrho_1 - 1, \varrho_2 - 1, \ldots, \varrho_l - 1$, and such that

$$f(z) = \sum_{k=1} P_k(z) \exp(\omega_k z) \not\equiv 0.$$

Put $n = \sum_{k=1}^{l} \varrho_k$ and $\Delta = \max_{k=1,\ldots,l} |\omega_k|$.

Let R, s, t be positive numbers, $s > 1$.

On applying theorem 4 of [16], we find for an arbitrary real number $\gamma > 1$

$$M(0, \gamma tR, f) \leqq \frac{\gamma^n - 1}{\gamma - 1} e^{tR\Delta(\gamma+1)} M(0, tR, f).$$

Putting $\gamma = (st+s+t)/t$ one has

$$\frac{\gamma^n - 1}{\gamma - 1} \leqq \frac{t}{st+s} \left(\frac{st+s+t}{t}\right)^n \leqq (1+s^{-1}) \left(\frac{st+s+t}{t}\right)^{n-1}.$$

Hence

$$M(0, (st+s+t)\,R, f) \leqq (1+s^{-1}) \left(\frac{st+s+t}{t}\right)^{n-1} e^{(st+s+2t)R\Delta} M(0, tR, f).$$

The combination of this inequality and lemma 1 yields

$$N(0, R, f) \leqq \frac{1}{\log s} \left\{(n-1) \log \left(\frac{st+s+t}{t}\right) + (st+s+2t)\,R\Delta + \frac{1}{s}\right\}$$

By the application of this inequality on $f(z-z_0)$ for an arbitrary complex number z_0, we obtain

Theorem 1. Let R, s, t be positive numbers, $s>1$, and let z_0 be an arbitrary complex number. Then, in the notation (2.1),

$$N(z_0, R, f) \leqq \frac{1}{\log s}\left\{(n-1)\log\frac{st+s+t}{t}+(st+s+2t)\,R\Delta+\frac{1}{s}\right\}. \tag{2.2}$$

This upper bound is better than both upper bounds given in theorem 5.1 of [15], p. 58.

Of course we are still free in our choice of s and t. Taking for example $s=5$, $t=\frac{1}{5}$ we obtain, since $N(z_0, R, f)=0$ for $n=1$:

Corollary.

$$N(z_0, R, f) \leqq 3(n-1)+4R\Delta. \tag{2.3}$$

It is striking that this upper bound depends on so few of the variables. It is independent both of the centre z_0 of the disk $C(z_0, R)$ and of the coefficients of the polynomials $P_1, P_2, \ldots, P_l$. Moreover, the dependence on the degrees of the polynomials P_k is expressed as a function of only one variable, n, and the dependence on the ω_k's also as a function of only one variable, Δ.

Upper bounds for the number, N_L, of zeros of the general exponential polynomial f in a square of side length L were previously given by S. Dancs and P. Turán, [3]. Their estimate was

$$N_L \leqq 9p^2\log(2lp)+5lp+L\Delta\left(5+\frac{p}{10l}\right)+p(2l+p)\log\left(1+\frac{5l^2p}{L\delta}\right), \tag{2.4}$$

with $p=\max\limits_{k=1,\ldots,l}\varrho_k$ and $\delta=\min\limits_{\substack{k,j=1,\ldots,l\\ k\neq j}}|\omega_k-\omega_j|$.

Every square of side length L is contained in a closed disk of radius $(3L/4)>(L/\sqrt{2})$. Hence the above corollary implies

$$N_L \leqq 3(n-1)+3L\Delta. \tag{2.5}$$

Since $n\leqq lp$, this estimate is considerably better than (2.4), and moreover it has the advantage that it is independent of δ.

3. In this section we shall investigate the behaviour of $N(z_0, R, f)$ in the case that $R\Delta$ is small with respect to n. It is well-known (see e.g. [7] theorem 1.2.1 on p. 205), that f can have a zero of the order $n-1$ in the centre of the disk. We therefore can not expect an estimate better than

$$N(z_0, R, f) \leqq n-1. \tag{3.1}$$

On the other hand, it follows from the same theorem that f cannot have zeros of order greater than $n-1$.

We have by theorem 1, since $\log \frac{st+s+t}{st} < \frac{s+t}{st}$

$$N(z_0, R, f) - (n-1) \leqq \frac{1}{\log s} \left\{ (n-1) \left(\frac{1}{s} + \frac{1}{t} \right) + (st+s+2t) R\Delta + \frac{1}{s} \right\}. \tag{3.2}$$

Assume

$$n \geqq 24 \text{ and } R\Delta \leqq n^{-2}. \tag{3.3}$$

Put $s = n-1$, $t = n-2$. Hence

$$N(z_0, R, f) - (n-1) \leqq \frac{1}{\log (n-1)} \left(3 + \frac{2}{n-2} \right) < 1.$$

Since $N(z_0, R, f) - (n-1)$ is an integer, we obtain

$$N(z_0, R, f) \leqq n-1.$$

This inequality is best possible in view of (3.1). So we have proved

Theorem 2. Let R be a real number, $R > 0$, and z_0 a complex number. Assume that in the notation (2.1) the inequalities $n \geqq 24$ and $R\Delta \leqq n^{-2}$ hold. Then

$$N(z_0, R, f) \leqq n-1.$$

Now we shall prove

Theorem 3. Let R be a real number, $R > 0$, and z_0 a complex number. Assume that in the notation (2.1) the inequality $n > R\Delta$ holds. Then

$$\frac{N(z_0, R, f)}{n} \leqq 1 + 6 \log^{-1} \frac{n}{R\Delta}.$$

Proof. We have by (3.2)

$$N(z_0, R, f) \leqq n + \frac{1}{\log s} \left\{ n \left(\frac{1}{s} + \frac{1}{t} \right) + (st+s+2t) R\Delta \right\}.$$

Put $s = n/R\Delta$, $t = 1$. Then

$$N(z_0, R, f) \leqq n + 3(n + R\Delta) \log^{-1} \frac{n}{R\Delta} < n \left(1 + 6 \log^{-1} \frac{n}{R\Delta} \right).$$

Corollary. Let $\varepsilon > 0$. Suppose $\frac{R\Delta}{n} < e^{-6/\varepsilon}$.

Then

$$N(z_0, R, f) \leqq n(1+\varepsilon).$$

This corollary is an improvement of lemma III, § 4, III of the book "Transcendental and algebraic numbers" of A. O. Gelfond, [6] pp. 140–141. I hope to return to applications of these results in the theory of transcendental numbers in a forthcoming paper.

4. We now consider the case that R satisfies the inequality

$$R\Delta \geqq n/e. \tag{4.1}$$

By theorem 1 we have, putting $s=e$ and $t=n(R\Delta(e+2))^{-1}$:

$$N(z_0, R, f) \leqq (n-1)\log\left(e+1+\frac{eR\Delta(e+2)}{n}\right)+R\Delta\left(\frac{n}{R\Delta}+e\right)+\frac{1}{e}\leqq$$

$$\leqq eR\Delta+n+\frac{1}{e}+(n-1)\log(e+2)+(n-1)\log\left(1+\frac{eR\Delta}{n}\right)\leqq$$

$$\leqq eR\Delta+3n+n\log\frac{eR\Delta}{n}+\frac{n^2}{eR\Delta}.$$

By (4.1) this implies

Theorem 4. Let us use the notation (2.1) and let R be a positive number such that

$$R\Delta \geqq n/e.$$

Then for an arbitrary number z_0

$$N(z_0, R, f) \leqq eR\Delta + n\left(5+\log\frac{R\Delta}{n}\right).$$

The result of this theorem implies that for fixed z_0

$$\overline{\lim_{R\to\infty}}\frac{N(z_0, R, f)}{R}\leqq e\Delta. \tag{4.2}$$

This upper bound cannot be improved starting from the formulas given in theorem 1; for

$$\frac{R\Delta(st+s+2t)}{\log s}\geqq\frac{sR\Delta}{\log s}\geqq eR\Delta.$$

(For $s>0$ the function $s/\log s$ has a minimum e for $s=e$). However, the inequality (4.2) is much weaker than an old result due to G. Pólya, [10], and E. Schwengeler, [13]. They gave a beautiful insight in the distribution of the zeros of f. It follows from their results that

$$N(z_0, R, f) = \theta R + 0(\log R), \qquad R\to\infty,$$

where $2\pi\theta$ equals the circumference of the smallest convex polygon containing the points $\omega_1, \omega_2, \ldots, \omega_l$. In 1965 D. C. Dickson, [4], proved an assertion of G. Pólya's, [11] p. 594, viz.

$$N(z_0, R, f) = \theta R + 0(1), \qquad R\to\infty.$$

However, it seems difficult to use their methods of proof in order to obtain upper bounds for $N(z_0, R, f)$ if R is finite.

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